A Variational-Fixed Point Iterative technique for the Solution of Second Order Differential Equations

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ABSTRACT   In this paper, we propose a variational-fixed point iteration technique for the solution of second order linear differential equations with two-point boundary value problems. The proposed method is endowed with the Galerkin method for the determination of the starting function. The numerical results show the validity and efficiency of the proposed method in comparison withthe exact solution and other existing methods.

(Keywords: Variational iteration method, Galerkin method, Fixed point iteration method, Differential equation)

INTRODUCTION

In the rapid development and growth of the sciences, problems mostly occur as linear and nonlinear differential equations. Many scientists are involved in the attempt to come out with methods with better approximations to the solutions of the given problems that are close to analytic solution (Araghi F. M. A., 2011). Many methods have been developed and others are still being developed. Some of these developed methods have their own limitations which have to undergo changes either to shed more light on their behavior or improve on their accuracy (Araghi F. M. A., 2011; He J. H., 2012; Kilicman A.,Wadai M., 2016). Few, among many methods developed, are the variational iteration method, the fixed point iteration method and so on. These methods have been proved by many authors to be one of the mathematical tools for solving linear and nonlinear differential equations (Araghi F. M. A., 2011; Lu J., 2007). Researchers have shown that the variational iteration method is flexible with good convergence property (He J. H.,2000). However, the success of the method depends on the careful choice of the starting function which is usually done by guessing (Daniel T., et al., 2012; Wu G. C., 2013; Kilicman A.,Wadai M., 2016; Lu J., 2007. In this study, we propose a Variational-Fixed Point Iterative technique where the starting function is determined by the use of the Galerkin approximation. The presentation of this article is as follows: In Section 2 we present the methodology, Section 3 deals with the numerical results and finally discussion and conclusion are given in the last section.

METHODOLOGY

The variational iteration method can be illustrated by considering differential equation LU +NU = g(s), where L and N are linear and nonlinear operator respectively while g(s) is a forcing term. According to variational iteration method, a correctional functional can be constructed as follows:

$$t_{n+1}(x) = t_n(x) + \lambda \int_0^x (LU_n(s) + NU_n(s) - g(s))ds \qquad [1]$$

where λ is a Lagrange multiplier, which can be identified optimally via variational theory, (He J. H., 2000; He J. H., 2007; Lu J., 2007; Noor M. A. & Mohyuddin S.T., 2008), the subscripts denotes the nth iteration (He J.H., 2008; He J.H., 2006), u_n is considered as a restricted variation. i.e $dU_n= 0$, also a simple formula has been developed where λ can easily be identified as

$$\lambda = \frac{(-1)^m}{(m-1)!}(t - x)^{m-1} \qquad [2]$$

to achieve convergence of result (Wu G. C., 2013). In order to overcome the difficulty of choosing the starting function $t_n(x)$ at n = 0 we employ the use of Galerkin method so as to have better choice of initial value to enhance convergence, which is illustrated as follows:

$$U(x) = U_o(x) + C_1 U_1(x) + C_2 U_2(x) + \cdots + C_n U_n(x) \qquad [3]$$

which is called the basis function where $C_1, C_2, \ldots, C_n$ are constants to be determined using the Galerkin

process (Gerald F. C., 2004). After single iteration of variational iteration method we obtain $t_1(x)$ which becomes the starting value of the fixed point iterative method; it is simple and easy to manipulate so that the process will be repeated until convergence occurs or the iteration is terminated, and thus

$$y_{n+1}(x) = (1 - \propto_n)y_n + \propto_n Ty_n$$
$$y'_{n+1}(x) = (1 - \propto_n)y'_n + \propto_n Ty'_n$$
$$\vdots$$
$$y''_{n+1}(x) = (1 - \propto_n)y''_n + \propto_n Ty''_n.$$

NUMERICAL EXPERIMENTS

Three experiments are considered to demonstrate our method.

Experiment 1. Consider
$$y'' - y = -2x^2, 0 \le x \le 1 \qquad [4]$$
subject to the boundary conditions
$$y(0) = 0, y(1) = 6. \qquad [5]$$
The analytic solution is given by

$$y_E(x) = 0.626070571e^x - 0.626070571e^{-x} + 2x^2 + 4. \qquad [6]$$

We apply the Galerkin Method to solve the problem. Let

$$U(x) = 6x + C_1(x - x^2) + C_2(x - x^3). \qquad [7]$$

After differentiating eq (7) twice and then substitute in eq (4) we get

$$-2C_1 - 6C_2(x) - 6x - C_1(x - x^2) + C_2(x - x^3) = -2x^2 \qquad [8]$$

Let

$$R(x, C_1, C_2) = -2C_1 - 6C_2(x) - 6x - C_1(x - x^2) + C_2(x - x^3) + 2x^2 \qquad [9]$$

be the residual. Then apply the Galerkin processes as follow

$$\int_0^1 (x - x^2)(R(x, C_1, C_2)) = 0$$
$$\int_0^1 (x - x^3)(R(x, C_1, C_2)) = 0.$$

Then solving both integral equations to obtain
$$C_1 = -\frac{54}{473}, C_2 = -\frac{28}{43}. \qquad [10]$$
Substitute eq(10) into eq(7) to get

$$U(x) = \left(\frac{2476}{473}\right)x + \left(\frac{54}{473}\right)x^2 + \left(\frac{28}{43}\right)x^3. \qquad [11]$$
By evaluating eq(11) at x = 0.5 we obtain

$$U(0.5) = 2.727272727 \quad . \qquad [12]$$

Now we will apply Variational Iteration Method as follows. Let us construct correctional functional

$$t_{n+1}(x) = t_n(x) + \lambda \int_0^x (t''_n(s) - t_n(s) + 2x^2(s))ds \qquad [13]$$

Let us assume our initial guess

$$t_0 = Ax + B. \qquad [14]$$

We differentiate eq (14) twice and substitute into eq(13) at n =0, $\lambda = (s - x)$ to get

$$t_1(x) = Ax + B - \frac{1}{2(-(Ax+B)+2x^2)x^2} \qquad [15]$$

By imposing the boundary conditions in eq(15), we obtain A = 14/3, B = 0. Substitute these values back in eq(15) to get

$$t_1(x) = \left(\frac{14}{3}\right)x - \frac{1}{2\left(-\left(\frac{14}{3}\right)x+2x^2\right)x^2} \qquad [16]$$

Then we evaluate eq(16) where x = 0.5 to get

$$t_1(0.5) = 2.562500000. \qquad [17]$$

Now we apply the variational-fixed point iteration technique as follows, that is, we use the Galerkin method as our starting values. Let

$$U(x) = 6x + C_1(x - x^2) + C_2(x - x^3) \qquad [18]$$

where C_1 and C_2 are constants to be determine using the Galerkin Method. After differentiating eq(18) twice and then substitute into eq(4) we get

$$C_1 = -\frac{54}{473}, C_2 = -\frac{28}{43} \qquad [19]$$
Then substitute eq(19) into eq(18) to get

$$U(x) = \left(\frac{2476}{473}\right)x + \left(\frac{54}{473}\right)x^2 + \left(\frac{28}{43}\right)x^3 \qquad [20]$$

We now apply the variational iteration method using eq (20) as a starting value which we will then eventually obtain

$$t_1(x) = \left(\frac{2476}{473}\right)x + \left(\frac{54}{473}\right)x^2 + \left(\frac{28}{43}\right)x^3 - \frac{1}{2}\left(\frac{108}{473} - \left(\frac{628}{473}\right)x + \left(\frac{892}{473}\right)x^2 - \left(\frac{28}{43}\right)x^3\right) \qquad [21]$$

then we evaluate eq(21) at x = 0.5 to get
$$t_1(0.5) = 2.732954545. \qquad [22]$$

We then apply the new method by using t1(x) as starting value as follows. Let

$$y''_{n+1}(x) = y_n(x) - 2x^2, n = 0,1,2,. \qquad [23]$$

At n = 0 we have

$$y''_1(x) = y_0(x) - 2x^2 \qquad [24]$$

where

$$y_0(x) = t_1(x) \qquad [25]$$

Substitute eq (25) into eq (24) to get

$$y''_1(x) = \left(\frac{2476}{473}\right)x + \left(\frac{54}{473}\right)x^2 + \left(\frac{28}{43}\right)x^3$$
$$- ((\frac{1}{2})(\frac{108}{473} - \left(\frac{628}{473}\right)x + \left(\frac{892}{473}\right)x^2$$
$$- \left(\frac{28}{43}\right)x^3)x^2 \qquad [26]$$

We integrate eq(26) twice successively to get

$$y_1(x) = \left(\frac{1238}{1419}\right)x^3 - \left(\frac{1}{6}\right)x^4 + \left(\frac{311}{4730}\right)x^5$$
$$+ 1/129)x^7 - (223/7095)x^6 + C_1 x$$
$$+ C_2 \qquad [27]$$

imposing the boundary conditions for eq(27) we obtain

$$y_1(x) = \left(\frac{1238}{1419}\right)x^3 - \left(\frac{1}{6}\right)x^4 + \left(\frac{311}{4730}\right)x^5$$
$$+ (1/129)x^7 - (223/7095)x^6$$
$$+ (37264/7095)x \qquad [28]$$

we repeat the process successively up to 7th iteration to obtain

$$y_7(x) = (1/3113535308083200)x^{19} -$$
$$\left(\frac{233}{63090057558528000}\right)x^{18} + \left(\frac{311}{14020012790784000}\right)x^{17} -$$
$$\left(\frac{1}{5230697472000}\right)x^{16} + \left(\frac{619}{154632494016000}\right)x^{15} -$$
$$\left(\frac{1}{21794572800}\right)x^{14} + (2329/2761294536000)x^{13} -$$
$$\left(\frac{1}{119750400}\right)x^{12} + (1341499/10195549056000)x^{11} -$$
$$\left(\frac{1}{907200}\right)x^{10} + \left(\frac{880361}{60825718800}\right)x^9 - \left(\frac{1}{10080}\right)x^8 +$$
$$\left(\frac{69060513719}{6627106886400}\right)x^7 - \left(\frac{1}{180}\right)x^6 + \left(\frac{10151896102421}{231948741024000}\right)x^5 -$$
$$\left(\frac{1}{6}\right)x^4 + \left(\frac{2510287020960113}{2867729889024000}\right)x^3 \left(\frac{2360924944683800977}{449516660104512000}\right)x \qquad [29]$$

then we evaluate y_7 at x = 0.5 to get

$$y_7(0.5) = 2.726362232 \qquad [30]$$

Experiment 2. Consider the problem

$$y'' - y = -2x; 0 \le x \le 1 \qquad [31]$$

subject to the boundary conditions

$$y(0) = 0, y(1) = 3. \qquad [32]$$

The analytic solution is given by

$$y_E(x) = \frac{e^1(e^x - e^{-x})}{e^2 - 1} + 2x. \qquad [33]$$

We apply the Galerkin method to solve the problem. Let

$$U(x) = 3x + C_1(x - x^2) + C_2(x - x^3) \qquad [34]$$

after differentiating eq(34) twice and then substitute in eq (31) we get

$$-2C_1 - 6C_2(x) - 3x - C_1(x - x^2) - C_2(x - x^3) = -2x \qquad [35]$$

and let

$$R(x, C_1, C_{2=}) = -2C_1 - 6C_2(x) - x - C_1(x - x^2) - C_2(x - x^3) \qquad [36]$$

be the residual. Then by applying Galerkin processes as follow

$$\int_0^1 (x - x^2)(R(x, C_1, C_2)) = 0$$

$$\int_0^1 (x - x^3)(R(x, C_1, C_2)) = 0.$$

Then solving both integral equations to obtain

$$C_1 = \frac{8}{473}, \quad C_2 = -\frac{7}{43} \qquad [37]$$

Substitute eq(37) into eq(34) to get

$$U(x) = \left(\frac{1350}{473}\right)x - (8/473)x + (7/43)x^3. \qquad [38]$$

Then we evaluate eq(38) at x = 0.5 to get

$$U(0.5) = 1.443181818. \qquad [39]$$

Now we will apply Variational Iteration Method as follows.
Let us construct correctional functional

$$t_{n+1}(x) = t_n(x) + \lambda \int_0^x (t''_n(s) - t_n(s) + 2x(s))ds. \qquad [40]$$

Let assume our initial guess

$$t_0(x) = Ax + B \qquad [41]$$

we differentiate eq(41) twice and substitute to eq (40) when n = 0, $\lambda = (s - x)$ to get

$$t_1(x) = Ax + B - 1/2(-(Ax + B) + 2x)x^2 \qquad [42]$$

Imposing the boundary conditions in eq(42), we obtain A = 8/3, B = 0. Substitute these values back in eq(42) to get

$$t_1(x) = \left(\frac{8}{3}\right)x + \left(\frac{1}{3}\right)x^3. \qquad [43]$$

Then we evaluate eq(43) where x = 0.5 to get

$$t_1(0.5) = 1.375000000. \qquad [44]$$

Now we apply the variational fixed point iteration method as follows. We use the Galerkin method as our starting values instead of guessing as follows. Let

$$U(x) = 3x + C_1(x - x^2) + C_2(x - x^3) \qquad [45]$$

where C_1 and C_2 are constants to be determine using Galerkin Method. After differentiating eq (45) twice and then substitute in eq(31) we get

$$C_1 = \frac{8}{473}, \ C_2 = -\frac{7}{43}. \qquad [46]$$

Substitute eq(46) into eq(45) to get

$$U(x) = \left(\frac{1350}{473}\right)x - (8/473)x + (7/43)x^3 \qquad [47]$$

We now apply the variational iteration method using eq(43)as a starting value we eventually obtain

$$t_1(x) = \left(\frac{1350}{473}\right)x - (8/473)x + (7/43)x^3$$
$$- (1/2(-\frac{16}{473} + \left(\frac{58}{473}\right)x$$
$$+ \left(\frac{8}{473}\right)x^2 - \left(\frac{7}{43}\right)x^3)x^2 \qquad [48]$$

We evaluate eq(48) at x = 0.5 to get
$$t_1(0.5) = 1.441761363. \qquad [49]$$

We then apply the new method by using $t_1(x)$ as starting value as follows. Let

$$y''_{n+1} = y_n(x) - 2x, n = 0,1,2,\ldots \qquad [50]$$
At n = 0 we have

$$y''_{n+1} = y_0(x) - 2x \qquad [51]$$
where
$$y_0(x) = t_1(x). \qquad [52]$$
Substitute eq(52) into eq(51) to get

$$t_1(x) = \left(\frac{404}{473}\right)x - (8/473)x + (7/43)x^3 -$$
$$(\frac{1}{2\left(-\frac{16}{473} + \left(\frac{58}{473}\right)x + \left(\frac{8}{473}\right)x^2 - \left(\frac{7}{43}\right)x^3\right)x^2} \qquad [53]$$

We integrate eq(53) twice successively to get

$$y_1(x) = \left(\frac{202}{1419}\right)x^3 + \left(\frac{12}{2365}\right)x^5 + \left(\frac{1}{516}\right)x^7 -$$
$$\left(\frac{2}{7095}\right)x^6 + C_1 x + C_2 \qquad [54]$$

And imposing the boundary conditions for eq(54) we obtain

$$y_1(x) = \left(\frac{202}{1419}\right)x^3 + \left(\frac{12}{2365}\right)x^5 + \left(\frac{1}{516}\right)x^7 -$$
$$\left(\frac{2}{7095}\right)x^6 + \left(\frac{80909}{28380}\right)x \qquad [55]$$

We repeat the process successively up to 3rd iteration to obtain

$$y_3(x) = \left(\frac{1}{37152}\right)x^9 - \left(\frac{1}{198660}\right)x^8 + \left(\frac{2}{16555}\right)x^7 +$$
$$\left(\frac{101}{14190}\right)x^5 + \left(\frac{24149}{170280}\right)x^3 +$$
$$(8155639/$$
$$2860704)x \qquad [56]$$

Then we evaluate y_3 at x = 0.5 to get

$$y_3(0.5) = 1.443410956 \qquad [57]$$

Experiment 3. Consider the problem
$$y'' - y' = -4x \ ; 0 \le x \le 1 \qquad [58]$$
subjects to the boundary conditions
$$y(0) = 0, y(1) = 3. \qquad [59]$$
The analytic solution is given by
$$y_E(x) = 1.745930121 - 1.745930121e^x + 4x + 2x^2. \qquad [60]$$
We apply the Galerkin method to solve the problem. Let

$$U(x) = 3x + C_1(x - x^2) + C_2(x - x^3). \qquad [61]$$

After differentiating eq(61) twice and substituting into eq(58) we get

$$-2C_1 - 6C_2 x - 3 - C_1(1 - 2x) - C_2(1 - 3x^2) = -4x \qquad [62]$$

Let
$$R(x, C_1, C_2) = -2C_1 - 6C_2(x) - 3 - C_1(1 - 2x) - C_2(1 - 3x^2) + 4x \qquad [63]$$

be the residual. Then by applying the Galerkin processes as follow

$$\int_0^1 (x - x^2)(R(x, C_1, C_2) = 0$$

$$\int_0^1 (x - x^3)(R(x, C_1, C_2)) = 0$$

and solving both integral equations, we obtain

$$C_1 = -\frac{77}{61}, \quad C_2 = \frac{30}{61}. \tag{64}$$

Substitute eq(64) into eq(61) to get

$$U(x) = \left(\frac{136}{61}\right)x - \left(\frac{77}{61}\right)x^2 + \left(\frac{30}{61}\right)x^3 \tag{65}$$

Then evaluate eq(65) at x = 0.5 to get

$$U(0.5) = 1.368852458 \tag{66}$$

Now, we apply the Variational Iteration Method as follows. We construct the correctional functional as

$$t_{n+1}(x) = t_n(x) + \lambda \int_0^x (t''_n(s) - t'_n(s) + 2x(s))ds. \tag{67}$$

Assume our initial guess to be

$$t_0(x) = Ax + B \tag{68}$$

By differentiating eq(68) twice and substituting into eq(67) atn = 0, $\lambda = (s - x)$ we get

$$t_1(x) = Ax + B - \left(\frac{1}{2}\right)(-A + 2x)x^2 \tag{69}$$

Imposing the boundary conditions in eq(69), we obtain A =8/3,B = 0. Substitute these values back in eq(69) to get

$$t_1(x) = \left(\frac{8}{3}\right)x - \left(\frac{1}{2}\right)\left(-\frac{8}{3} + 2x\right)x^2 \tag{70}$$

Then, we evaluate eq(70) where x = 0.5 to get

$$t_1(0.5) = 1.541666666 \tag{71}$$

Now, we apply the variational-fixed point iteration technique as follows. By using the Galerkin Method as our starting value, let

$$U(x) = 3x + C_1(x - x^2) + C_2(x - x^3) \tag{72}$$

where C_1 and C_2 are constants to be determined using the Galerkin method. After differentiating eq(72) twice and then substitute into eq(58) we get

$$C_1 = -\frac{77}{61}, \quad C_2 = \frac{30}{61}. \tag{73}$$

Then substitute eq(73) into eq(72) to get

$$U(x) = \left(\frac{136}{61}\right)x - (77/61)x^2 + (30/61)x^3 \tag{74}$$

Now, apply the variational iteration method using eq [74] as a starting value to obtain

$$t_1(x) = (136/61)x + (77/61)x^2 - (30/61)x^3 - \left(\left(\frac{1}{2}\right)\left(\frac{18}{61} - \left(\frac{90}{61}\right)x + \left(\frac{90}{61}\right)x^2\right)x^2\right) \tag{75}$$

Then evaluating eq(75) at x = 0.5 we get

$$t_1(0.5) = 1.378073769. \tag{76}$$

We then apply the new method by using $t_1(x)$ as starting value as follows. Let

$$y''_{n+1} = y'_n - 2x, n = 0,1,2,\ldots \tag{77}$$

At n = 0 e have

$$y''_1(x) = y'_0(x) - 2x \tag{78}$$

where

$$y_0(x) = t_1(x) \tag{79}$$

We differentiate eq(79) once and substitute it into eq [78] to get

$$y''_1(x) = (136/61) - (90/61)x - (90/61)x^2 - \left(\left(\frac{1}{2}\right)\left(\left(-\frac{90}{61}\right) + \left(\frac{180}{61}\right)x\right)x^2 - x\left(\frac{18}{16} - \left(\frac{90}{61}\right)x + \left(\frac{90}{61}\right)x^2\right)\right) \tag{80}$$

We integrate eq(80) twice successively to get

$$y_1(x) = \left(\frac{68}{61}\right)x^2 - \left(\frac{18}{61}\right)x^3 + \left(\frac{15}{244}\right)x^4 - \left(\frac{9}{61}\right)x^5 + C_1 x + C_2 \tag{81}$$

Imposing the boundary conditions for eq(81) we obtain

$$y_1(x) = \left(\frac{68}{61}\right)x^2 - \left(\frac{18}{61}\right)x^3 + \left(\frac{15}{244}\right)x^4 - \left(\frac{9}{61}\right)x^5 + \left(\frac{553}{244}\right)x. \tag{82}$$

We repeat the process successively up to 11th iteration to obtain

$$y_{11}(x) = -\left(\frac{1}{4923959040}\right)x^{14} + \left(\frac{1}{4220536320}\right)x^{13} - \left(\frac{1}{270547200}\right)x^{12} - \left(\frac{1}{22545600}\right)x^{11} - \left(\frac{47}{98380800}\right)x^{10} - \left(\frac{19}{3935232}\right)x^9 - \left(\frac{59639}{1377331200}\right)x^8 - \left(\frac{265}{765184}\right)x^7 - \left(\frac{14911}{6148800}\right)x^6 - \left(\frac{4294153}{295142400}\right)x^5 - \left(\frac{472356061}{6493132800}\right)x^4 - \left(\frac{202438499}{695692800}\right)x^3 + \left(\frac{11098475633}{98479180800}\right)x^2 + \left(\frac{1997810596051}{886312627200}\right)x. \tag{83}$$

Then we evaluate y11 at x = 0.5 to get

$$y_{11}(0.5) = 1.367377991.$$

RESULTS AND DISCUSSION

The results obtained from the experiments are tabulated with graphs as shown below. In the numerical experiments, accuracy, speed and the convergence of the method are of great importance. Accuracy measures the degree of closeness of the numerical solution of the theoretical solution while convergence is to measure the closeness of approach of successive iteration to the exact solution as the number of iteration increases.

Table 1, Table 2 and Table 3 showed the exact solution, the Galerkin method, the variational iteration method and the variational-fixed point iterative technique. These are compared and it is clear that the variational-fixed point iterative technique converges to the exact solution nicely, which indicate high convergence rate.

Note: $y_e(x)$ is the exact solution, $y_g(x)$ is the Galerkin method and $y_v(x)$ is the variational iteration method $y_n(x)$ is the new method; Table 1: Experiment 1, Table 2: Experiment 2, and Table 3: Experiment 3

Table 1. Experiment 1

x	y-e(x)	y_g(x)	y_v(x)	y_n(x)
0.1	0.526073237	0.5252600422	0.5246912050	0.5260732366
0.2	1.057178080	1.056710359	1.056050063	1.057178079
0.3	1.598029506	1.598257927	1.599060506	1.598029506
0.4	2.153039702	2.153809726	2.157224863	2.153039702
0.5	2.726362233	2.727272727	2.732954545	2.726362232
0.6	3.321933599	3.322553912	3.327960762	3.321933599
0.7	3.943512647	3.943560253	3.943645200	3.943512648
0.8	4.594718182	4.594198732	4.581490740	4.594718182
0.9	5.279065186	5.278376321	5.243452135	5.279065186
1.0	5.999999999	6.000000000	5.932346723	6.000000000

Table I shows the comparison with the Galerkin method, variational iteration method as well as variational-fixed point iteration technique. It is to be noted that only at the 5th iteration result was used in evaluating the approximate solution which converges sharply to the exact solution.

Table 2. Experiment 2

x	y_e(x)	y_g(x)	y_v(x)	y_n(x)
0.1	0.2852337034	0.2854059197	0.2855137104	0.2852339168
0.2	0.5713204546	0.5714503171	0.5716488795	0.5713208867
0.3	0.8591218385	0.8591099366	0.8591060360	0.8591225241
0.4	1.149516601	1.149361523	1.148760762	1.149517631
0.5	1.443409442	1.443181818	1.441761363	1.443410956
0.6	1.741740075	1.741547569	1.739626554	1.741742179
0.7	2.045492624	2.045435518	2.044343118	2.045495238
0.8	2.355705480	2.355822410	2.358463594	2.355708148
0.9	2.673481691	2.673684989	2.685203942	2.673483516
1.0	3.000000000	3.000000001	3.028541227	3.000000000

Table 2 shows the exact solution, the Galerkin method, variational iteration method and the variational-fixed point iteration method. It is to be noted that only 3rd iteration result was used which indicates a high order of convergence to the exact solution when compared.
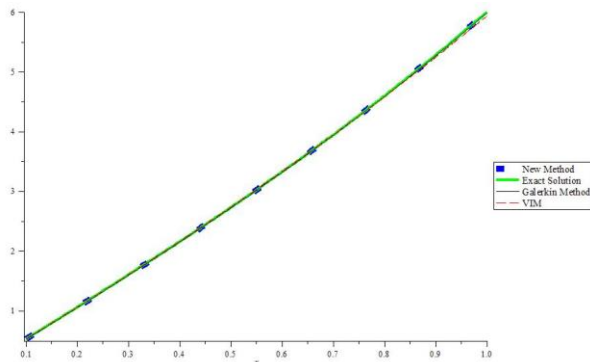
Table 3. Experiment 3

| x	y_e(x)	y_g(x)	y_v(x)	y_n(x)
0.1	0.2363789260	0.2350819672	0.2342704918	0.2363789261
0.2	0.4934462560	0.4924590164	0.4912786885	0.4934462549
0.3	0.7691709690	0.7691803279	0.7698442623	0.7691709685
0.4	1.061308446	1.062295082	1.067016393	1.061308444
0.5	1.367377993	1.368852458	1.378073769	1.367377991
0.6	1.684638024	1.685901640	1.696524591	1.684638022
0.7	2.010058614	2.010491803	2.014106557	2.010058612
0.8	2.340291179	2.339672131	2.320786885	2.340291178
0.9	2.671634964	2.670491803	2.604762295	2.671634964
1.0	3.000000000	3.000000000	2.852459016	3.000000000

Table 3 shows the exact solution, the Galerkin method, variational iteration method and the variational-fixed point iteration technique. It is also noted that only 11th iteration result was used in evaluating the approximate solution, which indicates a super convergence when compared with the exact solution.

CONCLUSION

From the experiments above, as the iteration increases, the error decreases and the rate of convergence of the proposed method increases, and it is also observed that the variational-fixed point iterative technique converges sharply to the exact solution which leads to super convergence. We conclude that the variational-fixed point iterative technique is a very strong and efficient technique for finding the approximate solution of second order linear boundary value problems because it has been proved to have an upper hand over the existing method. Thus we can conclude that the convergence of present method is faster than those in the literature.
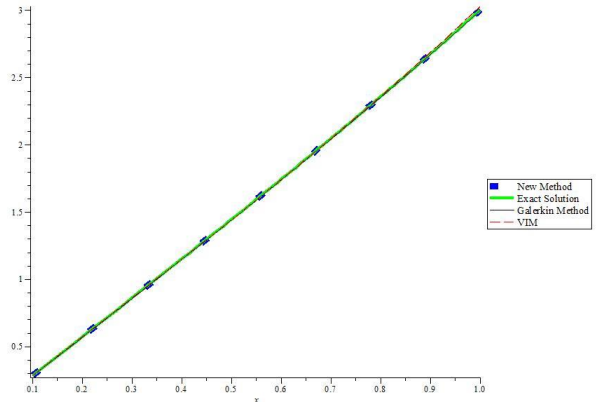
Figure 2. Experiment 2
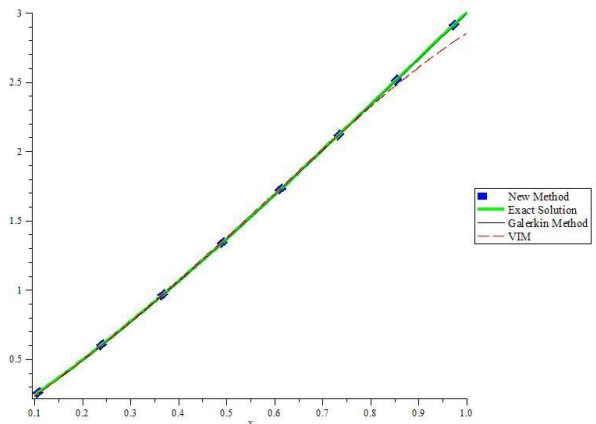
Figure 3. Experiment 3

Figure 1. Experiment 1

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