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A COMPARATIVE STUDY OF A CLASS OF LINEAR AND NONLINEAR PANTOGRAPH DIFFERENTIAL EQUATIONS VIA DIFFERENT ORTHOGONAL POLYNOMIAL WAVELETS

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Abstract: We propose a wavelet approach on different orthogonal polynomials for solving linear and nonlinear pantograph equations with stretch kind. The pantograph differential equation is a unique proportional delay functional differential equation class. It has been used to deal with numerous physics, mathematics, and engineering applications, such as quantum mechanics, control systems, electrodynamics, and number theory. This scheme is based on constructing the operational matrix for integration via different wavelets with their collocation nodes. This study aims to examine the numerical dynamics of the pantograph equation under stretch kind through different orthogonal polynomial wavelets. Illustrative examples are presented to highlight the flexibility of this scheme, and comparisons are made between the mentioned scheme and other existing schemes using tables and graphs. These numerical results correctly predict the applicability and effectiveness of the mentioned scheme.

Keywords: *Pantograph differential equation, muntz wavelets, chebyshev wavelets of different kinds, operational matrix, collocation nodes.*

1. Introduction

In variant mathematical modeling, delay differential equations (DEs) are key in solving various problems. Moreover, delay DEs are also used extensively in a distinct range of real-world situations such as economy, physiological and pharmaceutical kinetics, population dynamics, infectious diseases, chemical kinetics, epidemiology, ship navigational control, hydraulic network, etc. (Fox, 1971; Driver, 1977; Baker et al., 1995). Pantograph equation is a unique and special time delay DE that arises in several branches of applied and pure mathematics like number theory, quantum mechanics, dynamic systems, electrodynamics, control system, probability, and many more (Drfel and Iserles, 1997; Saadatmandi and Dehghan, 2009; Yusufoglu, 2010). In particular, Ockendon and Tayler (1971) and Tayler (1986) formulated this equation to describe how electricity is gathered through the pantograph of electrical locomotive. Figure 1 shows the pantograph model.

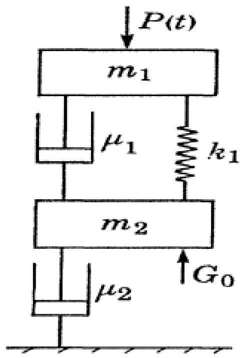

Figure 1. Pantograph Model (Ockendon and Tayler, 1971)

In this manuscript, we handle pantograph differential equation of stretch kind (PDESK) of the following form:

$$\frac{d}{dt}y(t) = \mathrm{g}(y(\lambda t),\, y(t),\, t), \quad 0 < \lambda \in \mathbb{R} < 1, \quad t \in [0,1],$$

with the condition

$$y(0) = r_0,$$

where r_0 is the real constant, and λ is a stretched argument. The given problem is an initial value problem. In general form, we can write the above problem as

$$G\left(\frac{d}{dt}y(t),\, y(\lambda t), y(t), t\right) = 0, \tag{1}$$

with the condition

$$y(0) = r_0, \tag{2}$$

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Various numerical approaches are based on existing orthogonal functions to solve the pantograph DEs. An overview of these approaches can be analyzed in the following studies: Sezer and Dascioglu (2007), Alomari et al. (2009), Yalcinbas et al. (2011), Sedaghat et al. (2012), Anakira et al. (2013), Tohidi et al. (2013), Yalcinbas et al. (2013), Bahsi and Cevik (2015), Jayadi et al. (2016), Yang (2018), Wang et al. (2019), Jafari et al. (2021), and Asma et al. (2022). In this study, we are interested in solving the PDESK defined in Eqs. (1-2) using wavelets based on different orthogonal functions.

In recent years, wavelets have become a growing and new area in physics, engineering, and mathematics. Wavelet analysis is a robust mathematical concept broadly used in image processing, signal processing, quantum field theory, numerical analysis, and several others (Daubechies, 1988; Mallat, 2018). Today, most physics models are analyzed through wavelet approaches. Due to the better precision of wavelets over other techniques, many researchers in different fields are interested in wavelets-based approaches (Rayal and Verma, 2020a; Rayal and Verma, 2020b; Rayal and Verma, 2020c; Rayal and Verma, 2022; Rayal et al., 2022; Rayal, 2023a; Rayal et al., 2023b). The most popular related techniques are the Legendre wavelets method (Hafshejani et al., 2011), Laguerre wavelets method (Shiralashetti et al., 2016), Hermite wavelets scheme (Saeed and Rehman, 2014), Bernoulli wavelets scheme (Rahimkhani et al., 2016), Gegenbauer wavelets method (Muhammad et al., 2017), Mamadu-Njoseh wavelets scheme (Rayal et al., 202, and Muntz wavelets scheme (Rayal, 2023d).

This study aims to compute the continuous approximate solutions of the PDESK defined in Eq. (1) using different orthogonal polynomial wavelets. An approximation scheme is introduced based on different orthogonal polynomial wavelets with integral operational matrix (IOM) and collocation grids to solve PDESK. The scheme converts the problems into simultaneous algebraic equations by expressing an unknown function y(t) in a truncated wavelet series. The wavelet characteristics, collocation technique, and integral operational matrix are utilized to evaluate y(t) in the given problem.

This manuscript is framed as follows: Section 2 introduces different orthogonal polynomial wavelets. Section 3 describes the function approximation through wavelets series. Section 4 explains the IOM for different wavelets. Section 5 proposes an approximate scheme for solving the problem. Section 6 estimates the errors to check the accuracy of the mentioned scheme. Section 7 contains examples of predicting the efficiency and precision of the proposed technique. Section 8 summarizes this study.

2. Orthogonal Polynomial Wavelets

This section defines the wavelets based on different orthogonal polynomials.

Muntz Wavelets

The definition of Muntz wavelets (MWs) on $[0,1)$ for $\gamma \in (0,1)$ is as follows (Bahmanpour, 2018):

$$\psi_{n,m}(t) = \begin{cases} \sqrt{\frac{1}{2} + m\gamma}\, 2^{\frac{k}{2}} P_m(2^{k-1}t - (n-1), \gamma), & \frac{n-1}{2^{k-1}} \le t < \frac{n}{2^{k-1}}, \\ 0, & elsewhere \end{cases}$$

where $= 0,1,2,3,\ldots,M-1$, $n = 1,2,3,\ldots,2^{k-1}$, k, M are natural numbers. The term $\sqrt{\frac{1}{2} + m\gamma}$ is employed for normality and $P_m(t)$ represents the Muntz functions of degree m that are orthogonal, corresponding to the unit weighted function $w(t)$ on $[0,1]$ and is represented in the following form:

$$P_m(t,\gamma) = \sum_{k=0}^{m} c_{m,k}\, t^{\gamma k},$$

where

$$c_{m,k} = \frac{(-1)^{m-k}}{\gamma^m k!\,(m-k)!} \prod_{i=0}^{m-1} ((k+i)\gamma + 1).$$

The MWs set is orthogonal under the weighted function, $w_n(t) = w(2^{k-1}t - n + 1)$.

Chebyshev Wavelets of The First Kind

The first kind of Chebyshev wavelets (CWs) have the arguments $\psi(n,m,k,t)$, in which $n = 1,2,3,\ldots,2^k$, $m = 0,1,2,\ldots,M-1$ is the order for first Chebyshev functions, $k \in \mathbb{N}$ and t represents the normalized time.

The definition of the CWs on $[0,1)$ is provided as (Tavassoli, 2009):

$$\psi_{n,m}(t) = \begin{cases} \frac{\alpha_m}{\sqrt{\pi}} 2^{\frac{k}{2}} T_m(2^{k+1}t - (2n-1)), & \frac{n-1}{2^k} \le t < \frac{n}{2^k}, \\ 0, & elsewhere \end{cases}$$

where

$$\alpha_m = \begin{cases} \sqrt{2} & m = 0 \\ 2 & m = 1,2,3,\dots . \end{cases}$$

Here, coefficient $\alpha_m/\sqrt{\pi}$ is employed for orthonormality and $T_m(t)$ is the first kind of Chebyshev function of degree m that is orthogonal under the weighted function $w(t) = 1/\sqrt{1-t^2}$ on $[-1,1]$ and has the following iterative relation:

$$T_0(t) = 1,$$

$$T_1(t) = t,$$

$$T_{m+1}(t) = 2t\, T_m(t) - T_{m-1}(t), \qquad m = 1,2,\dots .$$

The set of CWs is orthogonal under the weighted function, $w_n(t) = w(2^{k+1}t - 2n + 1)$.

Chebyshev Wavelets of the Second Kind

The second kind of Chebyshev wavelets (SCWs) have the arguments $\psi(n, m, k, t)$ in which $n = 1,2,3, \dots, 2^{k-1}$, $m = 0,1,2,3, \dots, M-1$ is the order for the second Chebyshev functions, $k \in \mathbb{N}$ and t represents the normalized time.

The definition of SCWs on $[0,1)$ is provided as (Zhu and Wang, 2013):

$$\psi_{n,m}(t) = \begin{cases} \sqrt{\frac{2}{\pi}}\, 2^{\frac{k}{2}} U_m(2^k t - 2n + 1), & \frac{n-1}{2^{k-1}} \le t < \frac{n}{2^{k-1}}, \\ 0, & elsewhere \end{cases}$$

Here, the term $\sqrt{2/\pi}$ is employed for normality and $U_m(t)$ is the second kind of Chebyshev function of degree m that is orthogonal under the weighted function $w(t) = \sqrt{1-t^2}$ on $[-1,1]$ and has the following iterative relation:

$$U_0(t) = 1,$$

$$U_1(t) = 2t,$$

$$U_{m+1}(t) = 2t\, U_m(t) - U_{m-1}(t), \qquad m = 1,2,3,\dots .$$

The set of SCWs is orthogonal under the weighted function. $w_n(t) = w(2^k t - 2n + 1)$.

Chebyshev Wavelets of the Third Kind

The third kind of Chebyshev wavelets (TCWs) have the arguments $\psi(n, m, k, t)$ in which $n = 1,2,3, \dots, 2^{k-1}$, $m = 0,1,2,3, \dots, M-1$ is the order for the third kind Chebyshev functions, $k \in \mathbb{N}$ and t represents the normalized time.

The definition of TCWs on $[0,1)$ is provided as (Polat, 2019):

$$\psi_{n,m}(t) = \begin{cases} \frac{1}{\sqrt{\pi}} 2^{\frac{k}{2}} V_m(2^k t - 2n + 1), & \frac{n-1}{2^{k-1}} \le t < \frac{n}{2^{k-1}}, \\ 0, & elsewhere \end{cases}$$

Here, the coefficient $\sqrt{1/\pi}$ is used for normality and $V_m(t)$ is the third Chebyshev function of degree m that is orthogonal under the weighted function $w(t) = \frac{\sqrt{1+t}}{\sqrt{1-t}}$ on $[-1,1]$ and has the following iterative relation:

$$V_0(t) = 1,$$

$$V_1(t) = 2t - 1,$$

$$V_{m+1}(t) = 2t\, V_m(t) - V_{m-1}(t), \qquad m = 1,2,3,\dots .$$

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The set of TCWs is orthogonal under the weighted function, $w_n(t) = w(2^k t - 2n + 1)$.

Chebyshev Wavelets of the Fourth Kind

The fourth kind of Chebyshev wavelets (FCWs) have the arguments $\psi(n, m, k, t)$ in which $n = 1,2,3, \ldots, 2^{k-1}$, $m = 0,1,2,3, \ldots, M-1$ is the order for the fourth kind Chebyshev functions, $k \in \mathbb{N}$ and t represents the normalized time.

The definition of FCWs on $[0,1)$ is as follows (Azodi and Yaghouti, 2018):

$$\psi_{n,m}(t) = \begin{cases} \frac{1}{\sqrt{\pi}} 2^{\frac{k}{2}} W_m(2^k t - (2n-1)), & \frac{n-1}{2^{k-1}} \leq t < \frac{n}{2^{k-1}}, \\ 0, & elsewhere \end{cases}$$

Here, the coefficient $1/\sqrt{\pi}$ is used for normality and $W_m(t)$ is the fourth Chebyshev function of degree m that is orthogonal under the weighted function $w(t) = \sqrt{\frac{1-t}{1+t}}$ on $[-1,1]$ and has the following iterative relation:

$$W_0(t) = 1,$$

$$W_1(t) = 2t + 1,$$

$$W_{m+1}(t) = 2t\, W_m(t) - W_{m-1}(t), \qquad m = 1,2,3, \ldots.$$

The set of CWs is orthogonal under the weighted function, $w_n(t) = w(2^k t - 2n + 1)$.

Now, the wavelet function approximation is described in the successive sections using the considered wavelet basis functions.

3. Function Approximation

A function $h(t)$ on $[0,1)$ can be approximated via considered wavelets as

$$h(t) \approx \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} e_{n,m}\, \psi_{n,m}(t), \tag{3}$$

where $e_{n,m}$ are computed by

$$e_{n,m} = \langle h(t), \psi_{n,m} \rangle_{w_n(t)} = \int_0^1 h(t) \psi_{n,m}(t) w_n(t) dt$$

Here, the notation $\langle ., . \rangle$ describes the inner product in $L^2[0,1]$ with the weighted function $w_n(t)$. The truncated form of Eq. (3) is rewritten as:

$$h(t) \approx \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} e_{n,m}\, \psi_{n,m}(t) = E^T \Psi(t) = \Psi^T(t)\, E, \tag{4}$$

where E and $\Psi(t)$ are provided by

$$\begin{aligned} E = & \left[e_{1,0}, \ldots, e_{1,(M-1)}, e_{2,0}, \ldots, e_{2,(M-1)}, \ldots, e_{2^{k-1},0}, \ldots, e_{2^{k-1},(M-1)}\right]^T \\ = & \left[e_1, e_2, \ldots, e_{\widehat{m}}\right]^T, \end{aligned} \tag{5}$$

$$\begin{aligned} \Psi(t) = & \begin{bmatrix} \psi_{1,0}(t), \ldots, \psi_{1,(M-1)}(t), \psi_{2,0}(t), \ldots, \psi_{2,(M-1)}(t), \ldots, \\ \psi_{2^{k-1},0}(t), \ldots, \psi_{2^{k-1},(M-1)}(t) \end{bmatrix}^T \\ = & \left[\psi_1, \psi_2, \ldots, \psi_{\widehat{m}}\right]^T. \end{aligned} \tag{6}$$

Here, $\widehat{m} = 2^{k-1} M$ denotes the total considered wavelets basis, but in the case of the first kind of Chebyshev wavelets $\widehat{m} = 2^k M$.

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4. Integral Operational Matrix for Orthogonal Polynomial Wavelets

This section provides the IOM $P_{\hat{m}\times\hat{m}}$ for different wavelets that play an important part in the PDESK solution. This operational matrix is employed to transform the given model to the algebraic system of equations in terms of wavelet coefficient. By applying the IOM, a large unknown coefficient vector does not occur when computing the numerical approximation of a linear and nonlinear PDESK class. Consequently, the calculations are made simple, resulting in better solution accuracy. In general,

$$\int_0^t \Psi(t)\,dt \approx P_{\hat{m}\times\hat{m}}\Psi(t), \tag{7}$$

where $\Psi(t)$ is provided in Eq. (6) and $P_{\hat{m}\times\hat{m}}$ is the IOM determined by

$$P_{\hat{m}\times\hat{m}} = \left\langle g_{\hat{m}\times 1}(t), \Psi^T_{\hat{m}\times 1}(t)\right\rangle_{w_n(t)},$$

where

$$g_{\hat{m}\times 1}(t) = \int_0^t \Psi(t)dt$$

and the notation $\langle .,. \rangle$ represents the inner product in $L^2[0,1]$ under the weighted function $w_n(t)$.

Using Eq. (7), construct the following IOM for different wavelets:

(a) The IOM of the Muntz wavelets ($\gamma = 0.5, k = 1, M = 8$):

$$(\mathrm{P}_{8\times 8})_{\mathrm{MWs}} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{2}}{5} & \frac{1}{10\sqrt{3}} & 0 & 0 & 0 & 0 & 0 \\ -\frac{\sqrt{2}}{5} & 0 & \frac{\sqrt{2}}{7\sqrt{3}} & \frac{\sqrt{2}}{35} & 0 & 0 & 0 & 0 \\ -\frac{1}{10\sqrt{3}} & -\frac{\sqrt{2}}{7\sqrt{3}} & 0 & \frac{2}{15\sqrt{3}} & \frac{\sqrt{5}}{42\sqrt{3}} & 0 & 0 & 0 \\ 0 & -\frac{\sqrt{2}}{35} & -\frac{2}{15\sqrt{3}} & 0 & \frac{2\sqrt{5}}{77} & \frac{\sqrt{2}}{33\sqrt{3}} & 0 & 0 \\ 0 & 0 & -\frac{\sqrt{5}}{42\sqrt{3}} & -\frac{2\sqrt{5}}{77} & 0 & \frac{\sqrt{10}}{39\sqrt{3}} & \frac{\sqrt{35}}{286} & 0 \\ 0 & 0 & 0 & -\frac{\sqrt{2}}{33\sqrt{3}} & -\frac{\sqrt{10}}{39\sqrt{3}} & 0 & \frac{\sqrt{14}}{55\sqrt{3}} & \frac{2}{65\sqrt{3}} \\ 0 & 0 & 0 & 0 & -\frac{\sqrt{35}}{286} & -\frac{\sqrt{14}}{55\sqrt{3}} & 0 & \frac{2\sqrt{14}}{221} \\ 0 & 0 & 0 & 0 & 0 & -\frac{2}{65\sqrt{3}} & -\frac{2\sqrt{14}}{221} & 0 \end{bmatrix};$$

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(b) The IOM of the first kind of Chebyshev wavelets ($k = 0, M = 8$):

$$\left(\mathrm{P}_{8\times 8}\right)_{\mathrm{CWs}} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{4\sqrt{2}} & 0 & \frac{1}{8} & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{3\sqrt{2}} & -\frac{1}{4} & 0 & \frac{1}{12} & 0 & 0 & 0 & 0 \\ \frac{1}{8\sqrt{2}} & 0 & -\frac{1}{8} & 0 & \frac{1}{16} & 0 & 0 & 0 \\ -\frac{1}{15\sqrt{2}} & 0 & 0 & -\frac{1}{12} & 0 & \frac{1}{20} & 0 & 0 \\ \frac{1}{24\sqrt{2}} & 0 & 0 & 0 & -\frac{1}{16} & 0 & \frac{1}{24} & 0 \\ -\frac{1}{35\sqrt{2}} & 0 & 0 & 0 & 0 & -\frac{1}{20} & 0 & \frac{1}{28} \\ \frac{1}{48\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{24} & 0 \end{bmatrix};$$

(c) The IOM of second kind Chebyshev wavelets ($k = 1, M = 8$):

$$\left(\mathrm{P}_{8\times 8}\right)_{\mathrm{SCWs}} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{3}{8} & 0 & \frac{1}{8} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{6} & -\frac{1}{12} & 0 & \frac{1}{12} & 0 & 0 & 0 & 0 \\ -\frac{1}{8} & 0 & -\frac{1}{16} & 0 & \frac{1}{16} & 0 & 0 & 0 \\ \frac{1}{10} & 0 & 0 & -\frac{1}{20} & 0 & \frac{1}{20} & 0 & 0 \\ -\frac{1}{12} & 0 & 0 & 0 & -\frac{1}{24} & 0 & \frac{1}{24} & 0 \\ \frac{1}{14} & 0 & 0 & 0 & 0 & -\frac{1}{28} & 0 & \frac{1}{28} \\ -\frac{1}{16} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{32} & 0 \end{bmatrix};$$

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(d) The IOM of third kind Chebyshev wavelets ($k = 1, M = 8$):

$$
\left(P_{8\times 8}\right)_{TCWs} = \begin{bmatrix}
\frac{3}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -\frac{1}{8} & \frac{1}{8} & 0 & 0 & 0 & 0 & 0 \\
\frac{5}{12} & -\frac{1}{8} & -\frac{1}{24} & \frac{1}{12} & 0 & 0 & 0 & 0 \\
-\frac{7}{24} & 0 & -\frac{1}{12} & -\frac{1}{48} & \frac{1}{16} & 0 & 0 & 0 \\
\frac{9}{10} & 0 & 0 & -\frac{1}{16} & -\frac{1}{80} & \frac{1}{20} & 0 & 0 \\
-\frac{11}{60} & 0 & 0 & 0 & -\frac{1}{20} & -\frac{1}{120} & \frac{1}{24} & 0 \\
\frac{13}{84} & 0 & 0 & 0 & 0 & -\frac{1}{24} & -\frac{1}{168} & \frac{1}{28} \\
-\frac{15}{112} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{28} & -\frac{1}{224}
\end{bmatrix};
$$

(e) The IOM of fourth kind Chebyshev wavelets ($k = 1, M = 8$):

$$
\left(P_{8\times 8}\right)_{FCWs} = \begin{bmatrix}
\frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{8} & \frac{1}{8} & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{12} & -\frac{1}{8} & \frac{1}{24} & \frac{1}{12} & 0 & 0 & 0 & 0 \\
\frac{1}{24} & 0 & -\frac{1}{12} & \frac{1}{48} & \frac{1}{16} & 0 & 0 & 0 \\
-\frac{1}{40} & 0 & 0 & -\frac{1}{16} & \frac{1}{80} & \frac{1}{20} & 0 & 0 \\
\frac{1}{60} & 0 & 0 & 0 & -\frac{1}{20} & \frac{1}{120} & \frac{1}{24} & 0 \\
-\frac{1}{84} & 0 & 0 & 0 & 0 & -\frac{1}{24} & \frac{1}{168} & \frac{1}{28} \\
\frac{1}{112} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{28} & \frac{1}{224}
\end{bmatrix};
$$

The next section explores the numerical PDESK solutions.

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5. Formulation of the Method

This section presents an approximate method based on orthogonal polynomial wavelets. The procedure of applying the method to a given problem is as follows.

Take the model equation from Eqs. (1-2) and expand the function, $\frac{d}{dt}y(t)$ via truncated series of wavelets over the interval [0,1) as

$$\frac{d}{dt}y(t) \approx E^T\Psi(t), \tag{8}$$

where E and $\Psi(t)$ are provided in Eqs. (5) and (6) respectively. By integrating Eq. (8) from 0 to t, we get

$$y(t) \approx E^T\int_0^t \Psi(t)\,dt + y(0) = E^T P_{\hat{m}\times\hat{m}}\Psi(t),$$

where $P_{\hat{m}\times\hat{m}}$ is the IOM of wavelets given in Eq. (7). After simplification, we obtain

$$\begin{aligned} y(t) \approx\ & E^T P_{\hat{m}\times\hat{m}}\Psi(t) + y(0) \\ =\ & E^T P_{\hat{m}\times\hat{m}}\Psi(t) + r_0 \\ =\ & E^T P_{\hat{m}\times\hat{m}}\Psi(t) + d^T\Psi(t) \\ =\ & (E^T P_{\hat{m}\times\hat{m}} + d^T)\Psi(t) = y_{\hat{m}}(t), \end{aligned} \tag{9}$$

where vector d is chosen as:

$$d^T\Psi(t) = r_0. \tag{10}$$

By using an approximation form of function $y(t)$ given in Eq. (9), we obtain $y(\lambda t)$ as

$$y(\lambda t) \approx (E^T P_{\hat{m}\times\hat{m}} + d^T)\Psi(\lambda t), \tag{11}$$

where $\Psi(\lambda t)$ is a stretched wavelets function. Using Eqs. (8-10) into Eq. (1), we obtain:

$$G(E^T\Psi(t),\, (E^T P_{\hat{m}\times\hat{m}} + d^T)\Psi(\lambda t),\, (E^T P_{\hat{m}\times\hat{m}} + d^T)\Psi(t),\, t) = 0. \tag{12}$$

Now, collocating the obtained system at the appropriate grids t_i:

$$G(E^T\Psi(t_i),\, (E^T P_{\hat{m}\times\hat{m}} + d^T)\Psi(\lambda t_i),\, (E^T P_{\hat{m}\times\hat{m}} + d^T)\Psi(t_i),\, t_i) = 0, \tag{13}$$

where

$$t_i = \frac{2i-1}{2^k M}, \quad i = 1,2,\dots,2^{k-1}M. \tag{14}$$

The resultant algebraic set in Eq. (13) can be evaluated properly for wavelet coefficients E. Finally, the solution $y_{\hat{m}}(t)$ of the given problem is achieved through the inclusion of estimated coefficient E into Eq. (9) as $y_{\hat{m}}(t) = (E^T P_{\hat{m}\times\hat{m}} + d^T)\Psi(t)$.

Figure 2 displays the flowchart for implementing the constructed scheme.

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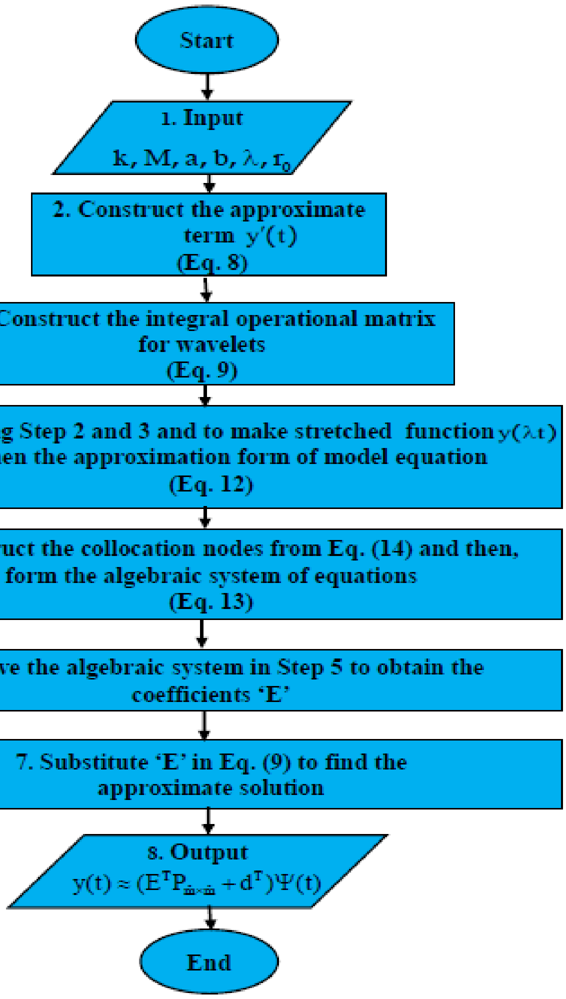

Figure 2. Flowchart for implementing the constructed scheme

6. Error Estimation of Solution

This section provides the convergence formulae to analyze the errors in the computation results. To investigate the accuracy of the proposed method, we define the error formulae as:

(a) Let $y_{\hat{m}}(t)$ be the estimate solution to $y(t)$ of Eqs. (1-2). Then $E_{Abs}(t)$ at $t \in [0,1]$ is calculated as

$$E_{Abs}(t) = |y(t) - y_{\hat{m}}(t)|,$$

where $y(t)$ is the analytical solution of the considered model.

(b) L_∞, the maximum absolute error is computed by

$$L_\infty = \max_{t\in[0,1]} |E_{Abs}(t)|$$

(c) The L^2 norm consecutive error $(\mathcal{C}.\mathcal{E})$ is computed by

$$\mathcal{C}.\mathcal{E} = ||y_{\hat{m}+1}(t) - y_{\hat{m}}(t)||_2, \quad t \in [0,1]. \tag{15}$$

(d) The reliability of the results and accuracy of the scheme can be checked through residual error function in the absence of an exact solution of the proposed model as:

$$E_{\hat{m}}(t) = \left|\frac{d}{dt} y_{\hat{m}}(t) - \mathrm{g}(y_{\hat{m}}(\lambda t),\, y_{\hat{m}}(t),\, t)\right|, \quad t \in [0,1]$$

If $E_{\hat{m}}(t) \to 0$ for $\hat{m}$, then the error decreases.

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7. Method Implementation

This section implements the constructed scheme with MWs, CWs, SCWs, TCWs, and FCWs to solve PDESK, and the approximated outputs obtained are compared with the corresponding available analytical solution. The L^2 norm consecutive errors and absolute errors demonstrate the accuracy of the constructed scheme. The proposed method is easy to implement, but the computational cost may be complex. All numerical outputs are computed using Mathematica.

Example 1.

Consider the PDESK (Bellen & Zennaro, 2003) as:

$$\frac{d}{dt}y(t) = y(0.5\,t), \quad 0 \le t \le 1,$$

with the condition

$$y(0) = 1$$

The closed-form solution of the considered example is provided (Bellen & Zennaro, 2003):

$$y(t) = \sum_{j=0}^{\infty} \frac{1}{j\,!}(2)^{\frac{j(1-j)}{2}} t^j.$$

We solve the above example for $\hat{m} = 8$ using the scheme introduced in Section 5. The wavelet coefficient vector of $y(t)$ can be determined as:

$$y_{\text{MWs}}(t) = 1.0 + 0.0000109285\sqrt{t} + 0.99986\,t + 0.000833015\,t^{1.5} + 0.24731t^2 + 0.0049748\,t^{2.5} + 0.0155824\,t^3 + 0.00292117\,t^{3.5}.$$

$$y_{\text{CWs}}(t) = 1.0 + t + 0.25t^2 + 0.0208333\,t^3 + 0.000651042\,t^4 + 8.13802 \times 10^{-6}\,t^5 + 4.23849 \times 10^{-8}\,t^6 + 9.49902 \times 10^{-11}t^7.$$

$$y_{\text{SCWs}}(t) = 1.0 + t + 0.25t^2 + 0.0208333\,t^3 + 0.000651042\,t^4 + 8.13802 \times 10^{-6}\,t^5 + 4.2385 \times 10^{-8}\,t^6 + 9.4948 \times 10^{-11}t^7.$$

$$y_{\text{TCWs}}(t) = 1.0 + t + 0.25t^2 + 0.0208333\,t^3 + 0.000651042\,t^4 + 8.13802 \times 10^{-6}\,t^5 + 4.23847 \times 10^{-8}\,t^6 + 9.50405 \times 10^{-11}t^7.$$

$$y_{\text{FCWs}}(t) = 1.0 + t + 0.25t^2 + 0.0208333\,t^3 + 0.000651042\,t^4 + 8.13802 \times 10^{-6}\,t^5 + 4.23854 \times 10^{-8}\,t^6 + 9.48487 \times 10^{-11}\,t^7.$$

Figures 3 and 4 display the achieved solutions and corresponding errors via different wavelets. Tables 1 and 2 present the approximate wavelet solutions through different wavelets with an exact solution and the Legendre wavelets method (LWM) (Hafshejani et al., 2011). One may observe from the tables and figures that the wavelet solutions converge faster to the analytical result. The error decreases more rapidly when the number of basic functions increases. Table 3 shows the L^2 norm consecutive error for the order of approximation $\hat{m} = 7{,}8$. Table 3 confirms that the error decreases with the increase of the order of approximation $\hat{m}$, which shows the accuracy of the described scheme. The L^2 norm consecutive error is calculated for the first time in this study using Eq. (15).

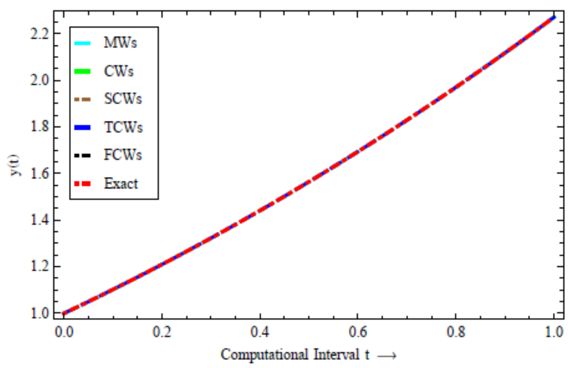

Figure 3. The behavior of the estimated solutions for different wavelets with $\hat{m} = 8$ in Example 1

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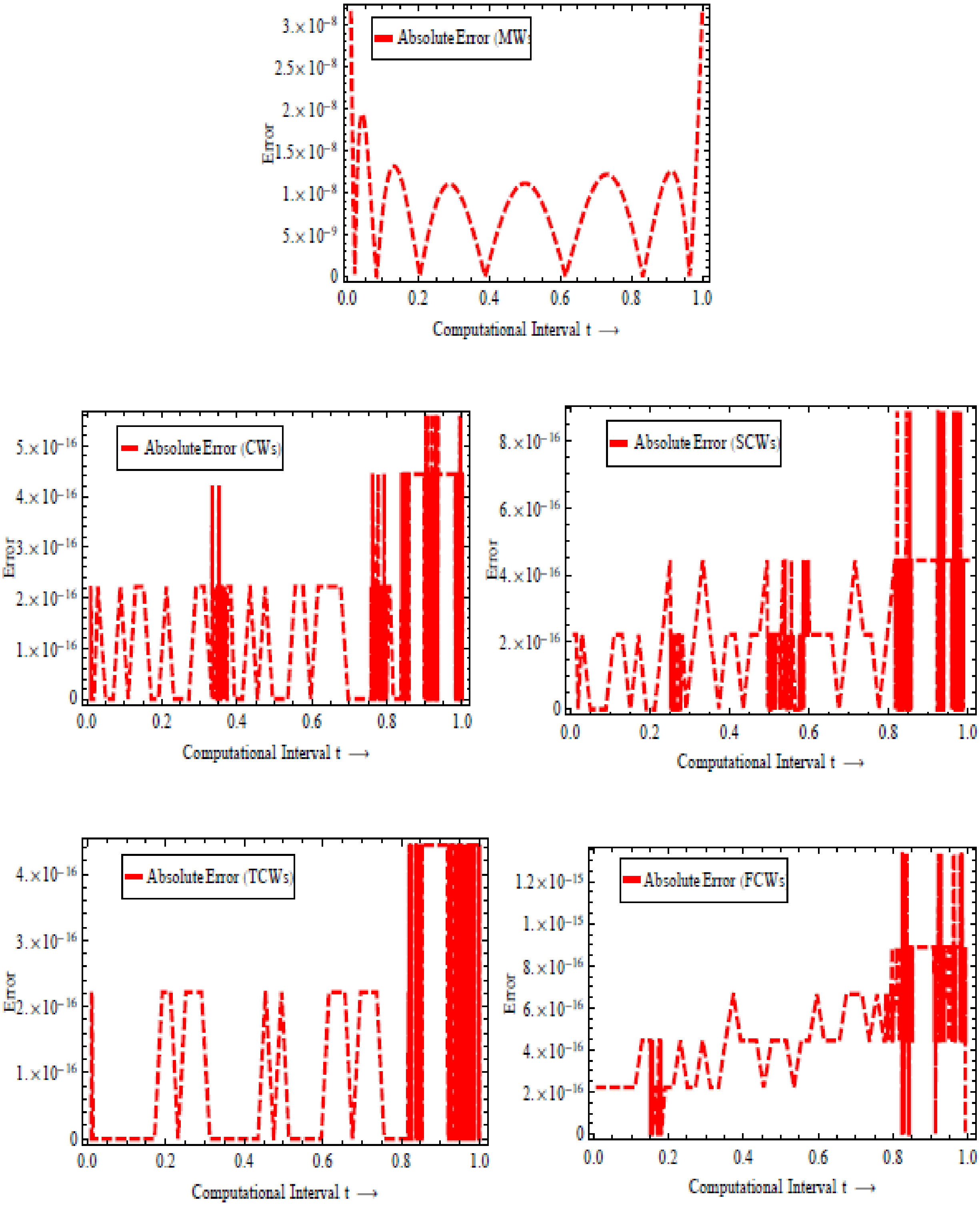

Figure 4. Absolute errors of the solutions via different wavelets with $\hat{m} = 8$ in Example 1

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Table 1. Computed values of $y(t)$ via the constructed approach compared to Example 1

t	LWs (Hafshejani et al., 2011), $\hat{m}=18$	Present Method (MWs), $\hat{m}=8$	Exact Solution
0.000	0.99999999999999	0.999999725312605	1.0000000000000000
0.125	1.12894709929840	1.128947112205302	1.1289470992984005
0.250	1.26595307192248	1.265953063336836	1.2659530719224836
0.375	1.41126781788344	1.411267815667616	1.4112678178834435
0.500	1.56514511174700	1.565145122864885	1.5651451117469977
0.625	1.72784263272750	1.727842631181262	1.7278426327275054
0.750	1.89962199489918	1.899621983149178	1.8996219948991855
0.875	2.08074877752466	2.080748786130356	2.0807487775246620
1.000	2.27149255550106	2.271492523809371	2.2714925555010614

Table 2. Approximated values of $y(t)$ via constructed scheme in Example 1

t	Present method (CWs), $\hat{m}=8$	Present method (SCWs), $\hat{m}=8$	Present method (TCWs), $\hat{m}=8$	Present method (FCWs), $\hat{m}=8$
0.000	1.0000000000000000	1.0000000000000000	0.9999999999999999	1.0000000000000000
0.125	1.1289470992984003	1.1289470992984005	1.1289470992984005	1.1289470992984003
0.250	1.2659530719224834	1.2659530719224836	1.2659530719224836	1.2659530719224834
0.375	1.4112678178834435	1.4112678178834437	1.4112678178834435	1.4112678178834432
0.500	1.5651451117469974	1.5651451117469979	1.5651451117469979	1.5651451117469972
0.625	1.7278426327275052	1.7278426327275054	1.7278426327275054	1.7278426327275047
0.750	1.8996219948991855	1.8996219948991860	1.8996219948991855	1.8996219948991848
0.875	2.0807487775246620	2.0807487775246620	2.0807487775246620	2.0807487775246614
1.000	2.2714925555010614	2.2714925555010620	2.2714925555010614	2.2714925555010610

Table 3. Efficiency of the constructed method in the terms of L^2 norm consecutive error via different wavelets in Example 1

$\hat{m}$	MWs	CWs	SCWs	TCWs	FCWs
7	5.86×10^{-6}	1.49×10^{-11}	1.50×10^{-11}	2.12×10^{-11}	2.03×10^{-11}
8	1.61×10^{-7}	8.25×10^{-15}	8.30×10^{-15}	1.18×10^{-14}	1.16×10^{-14}

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Example 2

Consider the linear PDESK (Yalcinbas, 2011; Bahsi & Cevik, 2015) as:

$$\frac{d}{dt}y(t) = -y(0.8\,t) - y(t), \quad 0 \le t \le 1,$$

with the condition

$$y(0) = 1.$$

There is no analytical solution to this problem. We treat this example for $\hat{m} = 8$ by using the scheme introduced in Section 5 and the wavelets series solution of $y(t)$ can be achieved as:

$$y_{\mathrm{MWs}}(t) = \begin{array}{l} 1.00007 - 0.0028539\sqrt{t} - 1.96514\,t - 0.193559\,t^{1.5} + 2.3507\,t^2 \\ -0.78919\,t^{2.5} - 0.59084\,t^3 + 0.29353\,t^{3.5}. \end{array}$$

$$y_{\mathrm{CWs}}(t) = \begin{array}{l} 1 - 2\,t + 1.79997\,t^2 - 0.983756\,t^3 + 0.370964\,t^4 - 0.10267\,t^5 \\ +0.0204596\,t^6 - 0.00229864\,t^7. \end{array}$$

$$y_{\mathrm{SCWs}}(t) = \begin{array}{l} 1 - 2\,t + 1.79996\,t^2 - 0.983709\,t^3 + 0.370873\,t^4 - 0.102583\,t^5 \\ +0.0204227\,t^6 - 0.00229449\,t^7. \end{array}$$

$$y_{\mathrm{TCWs}}(t) = \begin{array}{l} 1 - 2\,t + 1.79994\,t^2 - 0.9836\,t^3 + 0.370581\,t^4 - 0.102167\,t^5 \\ +0.0201234\,t^6 - 0.00220884\,t^7. \end{array}$$

$$y_{\mathrm{FCWs}}(t) = \begin{array}{l} 1 - 2\,t + 1.79998\,t^2 - 0.983827\,t^3 + 0.371189\,t^4 - 0.103032\,t^5 \\ +0.0207461\,t^6 - 0.00238706\,t^7. \end{array}$$

As mentioned above, we do not know the analytical solution to the given problem. Therefore, we estimate the solutions in Table 4 and observe a convergence. A comparison of Table 4 with the solutions achieved through several schemes is displayed in Table 5 (Tohidi et al., 2013; Yalcinbas et al., 2015; Yang, 2018; Yuzbas et al., 2014; Bahsi & Cevik, 2015; Yalcinbas et al., 2011; Sezer & Akyuz-Dascioglu, 2007). The calculated approximate solutions and corresponding absolute errors are displayed in Figures 5 and 6, respectively. Table 6 shows the L^2 norm consecutive error for the order of approximation $\hat{m} = 7,8$, which clearly shows the accuracy of the constructed approach. The numerical results of the suggested method are consistent.

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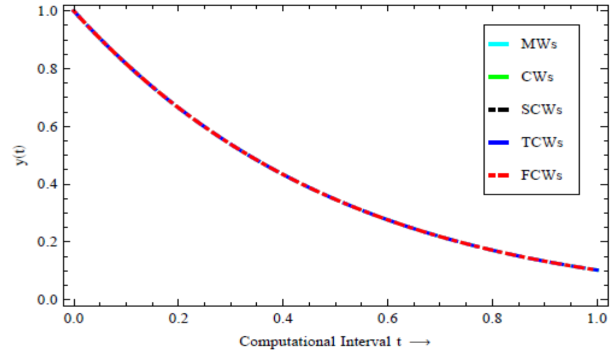

Figure 5. The behavior of the approximate wavelet solutions for different wavelets with $\hat{m} = 8$ in Example 2

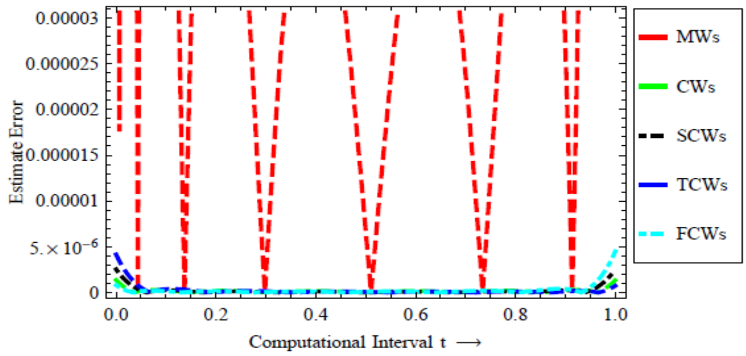

Figure 6. Estimate absolute errors of the solutions through different wavelets with $\hat{m} = 8$ in Example 2

Table 4. Approximated values of $y(t)$ with $\hat{m} = 8$ using the proposed scheme in Example 2

t	Present Method (MWs)	Present method (FCWs)	Present method (TCWs)	Present method (SCWs)	Present method (CWs)
0.0	1.0000737941	0.9999999944	0.9999999133	0.9999999523	0.9999999893
0.2	0.6646904337	0.6646910015	0.6646909898	0.6646909954	0.6646909970
0.4	0.4335604608	0.4335607737	0.4335607859	0.4335607800	0.4335607781
0.6	0.2764814550	0.2764823377	0.2764823264	0.2764823318	0.2764823311
0.8	0.1714859287	0.1714840995	0.1714841125	0.1714841063	0.1714841076
1.0	0.1026802165	0.1026700336	0.1026701212	0.1026700791	0.1026701151

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Table 5. Approximated values of $y(t)$ using different schemes for comparison in Example 2

t	Bernoulli method (Tohidi et al., 2013), $\hat{m} = 7$	Bernstein method (Yalcinbas et al. 2015), $\hat{m} = 11$	Chebyshev method (Yang, 2018), $\hat{m} = 7$	Laguerre method (Yuzbas et al., 2014), $\hat{m} = 9$	PIA (1,1) (Bahsi, & Cevik, 2015)	Hermite method (Yalcinbas et al., 2011), $\hat{m} = 9$	Taylor method (Sezer & Akyuz-Dascioglu, 2007), $\hat{m} = 12$
0.0	1.0000000	1.00000000	1.00000000	1.0000000	1.0000000	1.000000	1.000000
0.2	0.6646905	0.66469100	0.66469101	0.6646910	0.6646910	0.664691	0.664691
0.4	0.4335605	0.43356077	0.43356077	0.4335607	0.4335607	0.433561	0.433561
0.6	0.2764822	0.27648233	0.27648233	0.2764831	0.2764823	0.276482	0.276482
0.8	0.1714836	0.17148411	0.17148412	0.1714942	0.1714841	0.171484	0.171484
1.0	0.1026832	0.10267012	0.10267013	0.1027437	0.1026701	0.102670	0.102670

Table 6. Efficiency of the constructed method in terms of L^2 norm consecutive error using different wavelets in Example 2

$\hat{m}$	MWs	CWs	SCWs	TCWs	FCWs
7	1.10×10^{-4}	4.39×10^{-6}	4.13×10^{-6}	5.33×10^{-6}	6.31×10^{-6}
8	1.61×10^{-5}	2.01×10^{-7}	1.95×10^{-7}	2.56×10^{-7}	2.93×10^{-7}

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Example 3

Consider the PDESK as

$$\frac{d}{dt}y(t) = 0.95y(t) - y(0.99\,t), \quad 0 \leq t \leq 1,$$

with the initial condition

$$y(0) = 1.$$

There is no analytical solution to this problem. We solve it by considering the example for $\hat{m} = 8$ using the scheme introduced in Section 5. Because we do not know the analytical solution to the given problem, we show the accuracy of the described scheme by evaluating the residual error function. Table 7 shows the estimated numerical solutions via different wavelets, showing smooth convergence. Figure 7 plots the approximated solutions obtained for $\hat{m} = 8$. Figure 8 shows the graphical representation of the estimated errors in terms of residual function via different wavelets. Figure 7 shows that the approximated solution of the considered example decreases as t increases from 0 to 1. Table 8 exhibits the L^2 norm consecutive error for $\hat{m} = 7,8$. Table 8 confirms that the error decreases rapidly with increasing order of approximation, $\hat{m}$, which clearly shows the effectiveness of the constructed scheme.

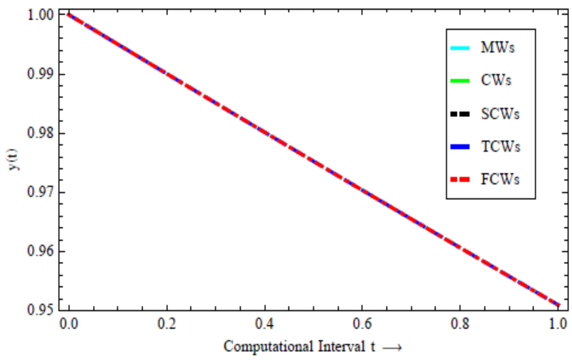

Figure 7. The behavior of the approximate solutions for different wavelets with $\hat{m} = 8$ in Example 3

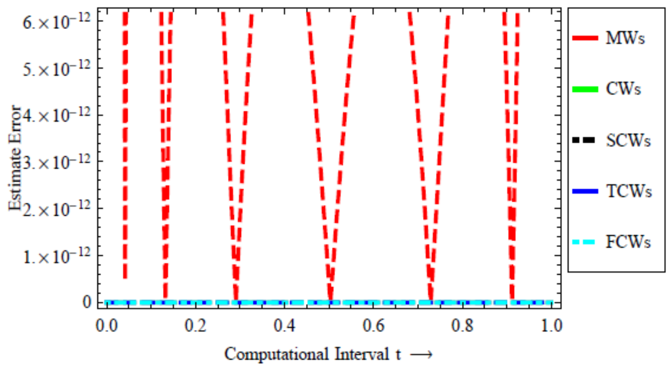

Figure 8. Error functions of the solutions using different wavelets with $\hat{m} = 8$ in Example 3

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Table 7. Approximated values of $y(t)$ using the proposed scheme in Example 3

t	Present Method (MWs), $\hat{m}=8$	Present method (CWs), $\hat{m}=8$	Present method (SCWs), $\hat{m}=8$	Present method (TCWs), $\hat{m}=8$	Present method (FCWs), $\hat{m}=8$
0.0	0.9999999999816	0.9999999999999	1.0000000000000	1.0000000000000	0.9999999999999
0.2	0.9900399198148	0.9900399198147	0.9900399198147	0.9900399198147	0.9900399198147
0.4	0.9801593591690	0.9801593591690	0.9801593591690	0.9801593591690	0.9801593591690
0.6	0.9703578393905	0.9703578393904	0.9703578393904	0.9703578393904	0.9703578393904
0.8	0.9606348837531	0.9606348837534	0.9606348837534	0.9606348837534	0.9606348837534
1.0	0.9509900174734	0.9509900174754	0.9509900174754	0.9509900174754	0.9509900174754

Table 8. Efficiency of the constructed method in terms of L^2, norm consecutive error using different wavelets in Example 3.

$\hat{m}$	MWs	CWs	SCWs	TCWs	FCWs
7	2.11×10^{-9}	2.62×10^{-16}	2.36×10^{-17}	2.73×10^{-17}	1.60×10^{-16}
8	1.17×10^{-11}	2.16×10^{-16}	3.30×10^{-18}	1.33×10^{-17}	3.11×10^{-18}

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Example 4.

Consider the nonlinear PDESK (Hafshejani et al., 2011; Anakira et al., 2013) as

$$\frac{d}{dt}y(t) = 1 - 2y^2\left(\frac{t}{2}\right), \quad 0 \le t \le 1,$$

with the condition

$$y(0) = 0.$$

The analytical solution to the example is provided as follows:

$$y(t) = sin(t).$$

We treat this example for $\widehat{m} = 8$ using the procedure given in Section 5. Figure 9 plots the approximate solutions obtained using different wavelets for $\widehat{m} = 8$. Table 9 displays the estimated absolute errors using different wavelets to compare the method (Hafshejani et al., 2011). Table 10 gives the maximum absolute error for $\widehat{m} = 6{,}7{,}8$. The maximum absolute errors to the same problem are 1.2×10^{-6}, 4.0×10^{-8}, 9.99×10^{-10}, and 1.2×10^{-7} (for 3 iteration), respectively (Alomari et al., 2009; Anakira et al., 2013; Hafshejani et al., 2011; Bahsi & Cevik, 2015). Table 11 exhibits the L^2 norm consecutive error for $\widehat{m} = 7{,}8$, which confirms that the error decreases rapidly as the order of approximation $\widehat{m}$ increases.

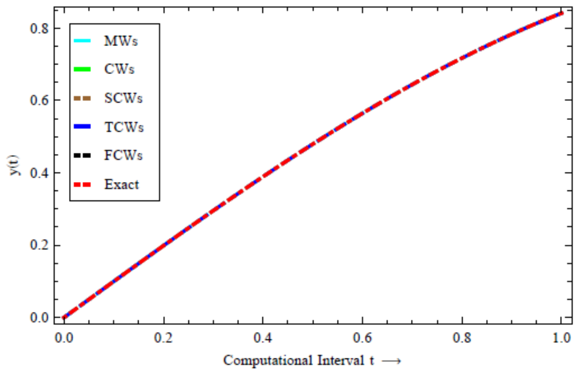

Figure 9. The behavior of the approximate solutions for different wavelets with $\widehat{m} = 8$ in Example 4

Table 9. Absolute errors of $y(t)$ using the current scheme at $\widehat{m} = 8$ compared to Example 4

t	MWs	CWs	SCWs	TCWs	FCWs	LWM (Hafshejani et al., 2011) $\widehat{m} = 18$
0.125	9.7×10^{-6}	5.1×10^{-10}	2.5×10^{-10}	1.1×10^{-10}	3.8×10^{-10}	1.9×10^{-9}
0.250	6.6×10^{-6}	1.4×10^{-11}	1.7×10^{-10}	3.6×10^{-10}	1.6×10^{-11}	1.9×10^{-9}
0.375	6.4×10^{-6}	2.8×10^{-11}	2.8×10^{-11}	1.3×10^{-10}	1.8×10^{-10}	1.9×10^{-9}
0.500	7.3×10^{-6}	4.7×10^{-10}	2.9×10^{-10}	2.8×10^{-10}	3.1×10^{-10}	9.9×10^{-10}
0.625	5.1×10^{-6}	2.8×10^{-11}	1.3×10^{-11}	1.2×10^{-10}	1.5×10^{-10}	9.9×10^{-10}
0.750	2.9×10^{-6}	2.6×10^{-11}	1.1×10^{-10}	8.3×10^{-11}	3.0×10^{-10}	9.9×10^{-10}
0.875	4.0×10^{-6}	4.3×10^{-10}	1.3×10^{-10}	2.8×10^{-10}	1.4×10^{-11}	9.9×10^{-10}
1.000	2.7×10^{-6}	5.6×10^{-10}	1.8×10^{-9}	4.1×10^{-10}	3.2×10^{-9}	9.9×10^{-10}

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Table 10. Maximum absolute error using different wavelets in Example 4

$\hat{m}$	MWs	CWs	SCWs	TCWs	FCWs
6	1.97×10^{-4}	2.96×10^{-7}	2.02×10^{-7}	1.36×10^{-7}	3.09×10^{-7}
7	2.59×10^{-5}	1.66×10^{-8}	7.53×10^{-9}	8.36×10^{-9}	1.30×10^{-8}
8	7.42×10^{-6}	2.20×10^{-10}	8.63×10^{-11}	3.35×10^{-10}	1.64×10^{-10}

Table 11. Efficiency of the proposed method in terms of L^2 norm consecutive error using different wavelets in Example 4

$\hat{m}$	MWs	CWs	SCWs	TCWs	FCWs
7	1.50×10^{-4}	2.34×10^{-7}	2.43×10^{-7}	3.46×10^{-7}	3.21×10^{-7}
8	1.22×10^{-5}	1.53×10^{-8}	1.50×10^{-8}	2.12×10^{-8}	2.11×10^{-8}

8. Conclusion

This paper proposes an approximation scheme using five orthogonal polynomial wavelets to solve PDESK. This method is examined using four problems. The error graphs and tables show the Chebyshev wavelets family, especially SCWs, is good for an approximate PDESK solution. Since most elements of derived matrices in the scheme are zeros, the computing time is short. The key advantage of the constructed scheme is that it can obtain results with high accuracy using fewer collocation nodes. The approximated PDESK solutions are provided in the form of graphs and tables. The obtained solution for the given examples shows that this scheme perfectly approximates the existing exact solution. The developed scheme is simple to implement.

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